ON THE ROLE OF MASS IN THE MATHEMATICAL STRUCTURE OF NEWTONIAN AND SPECIAL RELATIVISTIC MECHANICS

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Abstract

We consider five-dimensional real linear spaces with a (otherwise well-known) linear action of the Galilei and the Poincare group on them, describe the geometry of these two spaces, and show, that these geometries comprise the notions of space-time, mass, momentum, force and physical dimensions in a natural way. In this way we geometrize the quantity of mass and integrate it together with space-time into two geometries in a natural way, so that these geometries are perfectly suitable for underlying for the Newtonian and special relativistic mechanics of pointlike bodies.

I. INTRODUCTION

A classical mechanical theory begins usually with kinematics, i.e. with the definition and description of some geometrical notions underlying the theory, namely of space, time and motion.

The introduction of space-time by Einstein was a great step towards the proper understanding of the geometrical structure of space and time, and soon after that the well known nonrelativistic or Galilei space-time for Newtonian mechanics was also introduced by H. Weyl (see [1]). One can find the definition of this space-time in many modern books, e.g. [2] [3] [4].

It is clear, that it fits very well to Newtonian mechanics. It is very mysterious, however, that in order to be able to treat Newtonian dynamics (not some generalized version of it) we have to introduce essentially one (and only one) more quantity, namely the mass. It seems to us, that while the structure of space-time is very well understood, this is not the case with the step from kinematics to dynamics, although there are some well-known results about the mass of free particles within the framework of Hamiltonian mechanics and quantum theory. One usually says (in Newtonian dynamics) that the mass is some

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parameter. One also refers to fundamental experiments sometimes (e.g. [2]).
The main purpose of this paper is to offer an answer to this problem, i.e. the problem of the role of mass in the Newtonian and in the special relativistic mechanics of pointlike bodies.

Our answer is given as a reformulation of the elements of the two mechanical theories. We did neither aspire to treat the whole material of Newtonian mechanics and special relativity, nor did we want to provide an introduction. We restrict ourselves to those parts which we find necessary. We assume that the reader is familiar with Newtonian mechanics and special relativity and the abounding standard mathematical notions.

In section II. we treat Newtonian mechanics, in section III. special relativistic mechanics in an analogous manner, and in section IV. we deal with the relationship between the quantity which we call mass and the quantity which is called mass in the literature (in particular in the theorems about free particles and their relationship with the Galilei-group). In the appendix we introduce some operations and notions which are used throughout the text.

The fundamental object will be a geometrical structure denoted by (V, G) in the formulation of both mechanical theories. This is similar to a G-module, and consists of two parts:
– a *linear* space V
– a subgroup G of $GL(V)$, which we call symmetry group.

In particular, we shall have *five-dimensional linear* spaces, and the symmetry group will be the Galilei group and the Poincare group (whose action will be *linear*). We will see, that these spaces comprise the usual quantities of mass, space, time, momentum, force, etc. so we shall call them Newtonian and Einsteinian mechanical spaces. In this way mass will be geometrized, and the classical abyss between kinematics and dynamics will be lessened.

We will see, that it is possible to regard larger groups than the Galilei and the Poincare group as symmetry groups of mechanics, we will offer an answer to the question about the mathematical role or background of the fact that there are three independent dimensions (mass, length and period of time) in Newtonian mechanics (and two in spec.rel.), and we will also see that the two mechanical theories can be formulated without orientations (of space, time, mass).

In our presentation the group theoretical point of view is favoured. This will mean that we shall introduce various structures (so far as possible) as ones determined by the symmetry group (so far as possible) and we shall distinguish and appraise the various structures according to their invariance properties (i.e. relationship to the symmetry group). This point of view is common in modern physics (probably since Wigner), in geometry since the Erlangen programme and in a major part of present day mathematics as well (through the notion of category).

II. NEWTONIAN MECHANICS

1. Definition of the Newtonian space

`2.1 DEFINITION:`
We call the following real Lie group the Galilei group:

$$G^R = \left\{ \begin{pmatrix} O & v & x \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \in GL(R^3 \times R \times R) \mid O \in SO(3),\ x, v \in R^3,\ t \in R \right\}$$

Let V be a 5-dimensional real linear space, G a subgroup of $GL(V)$ so that there is a $b : V \to R^5$ linear isomorphism for which the map $i : GL(V) \to GL(R^5), \quad g \mapsto bgb^{-1}$ establishes a group isomorphism between G and G^R.
We call the pair (V, G) an (unoriented) Newtonian mechanical space, and we call a map b satisfying the former condition an inertial reference frame. Given two inertial reference frames b_1, b_2 we call $b_1 {b_2}^{-1}$ the map between the two reference frames. If we have a certain (V, G) specified, then we call also G Galilei group (or the Galilei group belonging to V). The elements of G act canonically on V by linear automorphisms $G \times V \to V, \quad (g, v) \mapsto g(v)$ so the elements of G are also called Galilei transformations. Obviously (R^5, G^R) is an example of Newtonian mechanical spaces which we call coordinate space.
We call two Newtonian mechanical spaces $(V_1, G_1), (V_2, G_2)$ isomorphic if there exists a $b : V_1 \to V_2$ linear isomorphism so that the map $i : GL(V_1) \to GL(V_2), \quad g \mapsto bgb^{-1}$ establishes a group isomorphism between G_1 and G_2. The Newtonian mechanical spaces constitute the objects of a category whose morphisms are the isomorphisms defined just now. All objects of this category are isomorphic. $\diamond$

G^R is a very well-known form of the Galilei group. The action of an element of G^R looks like this:

$$\begin{pmatrix} O & v & a \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} Ox + vy + az \\ y + tz \\ z \end{pmatrix}.$$

2. Properties of the Newtonian space

We describe some properties of the Newtonian mechanical spaces now. For this purpose we assume that we are given a certain Newtonian mechanical space (V, G), which we can also think of as the proper physical space.

`2.2. PROPOSITION:`
The group of automorphisms of (R^5, G^R) is a 13-dimensional real Lie group

$$\bar{G}^R = \left\{ \begin{pmatrix} A & a & b \\ 0 & d & c \\ 0 & 0 & e \end{pmatrix} \in GL(R^3 \times R \times R) \mid AA^T = n \cdot Id,\ n \in R^+,\ d, e \in R \setminus \{0\}, \right.$$

$$a,\ b \in R^3 \ \}$$

According to the definition the elements of $\bar{G}^R$ are the transformations between inertial reference frames.
Each element $\bar{g}$ of $\bar{G}^R$ can uniquely be written in the form

$$\bar{g} = cg,$$

where $c \in C^R$, $g \in G^R$, C^R is a subgroup of $\bar{G}^R$:

$$C^R = \left\{ \begin{pmatrix} a \cdot Id & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \in GL(R^3 \times R \times R \ ) | \ a, b, c \in R \setminus \{0\} \right\}.$$

G^R is an invariant subgroup of $\bar{G}^R$. C^R is not invariant, so $\bar{G}^R$ is the semidirect product of C^R and G^R. $\diamond$

`2.3. DEFINITION:`
Denoting the automorphism group of (V, G) by $\bar{G}$, let $C = \bar{G}/G$. $\diamond$

C is canonically isomorphic to C^R. C is not invariant subgroup of $\bar{G}$, but there is a class of conjugate subgroups of $\bar{G}$ isomorphic to C.

`2.4. PROPOSITION:`
The following subsets of $R^5 \equiv R^3 \times R \times R$ are invariant under the action of G^R :
— for all $m \in R$ the set $M^R_m = \{(x, y, z) \in R^5 \mid z = m\}$,
— for all $mt \in R$ the set $E^R_{mt} = \{(x, y, z) \in R^5 \mid y = mt,\ z = 0\}$,
— for all $md \in R^+_0$ the set $S^R_{md} = \{(x, y, z) \in R^5 \mid \sqrt{< x, x >} = md,\ y = z = 0\}$.
($<, >$ means the usual scalar product on R^3.)

The sets M^R_m, $m \in R$ are parallel 4-dimensional hyperplanes,
the sets E^R_{mt}, $mt \in R$ are parallel 3-dimensional hyperplanes in M^R_0,
the sets S^R_{md}, $md \in R^+_0$ are 2-dimensional similar spheres around the 0 contained in E^R_0.
(By 'similar' we mean that $S^R_x = (x/y) \cdot S^R_y$ for $x, y \in R, y \neq 0$.)

The orbits of the action of G^R are :
M^R_m , $m \in R \setminus \{0\}$
E^R_{mt} , $mt \in R \setminus \{0\}$
S^R_{md} , $md \in R^+_0$.

We see from this that V also decomposes to invariant hyperplanes and spheres uniquely in the way described above. By sphere we mean the level set of some positive definite quadratic form. This decomposition can be obtained by pulling back the one of R^5 to V with any arbitrarily chosen inertial reference frame. $\diamond$

Let us introduce the following notation for the components of V:
$M_{m(v)}$ denotes the 4-dimensional hyperplane which contains $v \in V$,
$E_{mt(v)}$ denotes the 3-dimensional hyperplane in M_0 which contains v, where $M_0 = M_{m(0)}$,
$S_{md(v)}$ denotes the sphere in E_0 which contains v, where $E_0 = E_{mt(0)}$.

`2.5. PROPOSITION:` The subgroup of $GL(V)$ which preserves the above decomposition of V is $\bar{G}$. The connected subgroup under whose action the components of the above decomposition are invariant is the Galilei group G. $\diamond$

This means, that the Newtonian mechanical space could have been defined in terms of the above decomposition.

Let us denote
the set of all 4-dimensional hyperplanes of the decomposition of V by $[kg]$,
the set of all 3-dimensional hyperplanes of the decomposition of V by $[kgs]$,
the set of all spheres of the decomposition of V by $[kgm]^+$.

The reason for this notation will be clear later. $[kg]$ has an 1-dimensional linear space structure : $[kg] \equiv V/M_0$. $[kgs] \equiv M_0/E_0$ is a 1-dimensional linear space as well and $[kgm]^+$ is the positive part of a 1-dimensional oriented linear space (multiplication with a real number can be defined through a representative and addition can be defined using two parallel representatives). Note, that $[kg]$ and $[kgs]$ are not oriented.

The group $\bar{G}$ acts on $[kg] \times [kgs] \times [kgm]$ by linear isomorphisms. The kernel of this action is the subgroup GP (a group with two topological components) of $\bar{G}$ under the action of which the components of V are invariant. (This group acts on E_0 effectively as an $O(3)$.) It is the (multiplicative) group $\bar{G}/GP \equiv R^+ \times (R \setminus \{0\}) \times (R \setminus \{0\})$ which acts effectively on $[kg] \times [kgs] \times [kgm]$. (Note, that $C \equiv (R \setminus \{0\}) \times (R \setminus \{0\}) \times (R \setminus \{0\})$ and $\bar{G}/GP \equiv C/(\{-1, 1\} \times \{1\} \times \{1\})$.)

`2.6. DEFINITION:`
We introduce the following ($\bar{G}$- equivariant) maps:

$m : V \to [kg],\ v \mapsto M_{m(v)}$
$mt : M_0 \to [kgs],\ v \mapsto E_{mt(v)}$
$md : E_0 \to [kgm]_0^+,\ v \mapsto S_{md(v)}.\diamond$

The first two ones are linear maps.

In the remaining part we need some operations with one-dimensional linear spaces. These are introduced in the appendix, so the reader is advised to read it before further advance.

It is clear, that md determines a positive definite bilinear map

$$<,>:\ E_0 \times E_0 \to [kgm]^2, \quad (v, w) \mapsto 1/4 \cdot ((md(v + w))^2 - (md(v - w))^2),$$

which we call the (generalized) Euclidean scalar product, which also determines a tensor $g \in Sym^2(E_0^*) \otimes [kgm]^2 \equiv Hom(Sym^2(E_0), [kgm]^2)$.

The situation is the following now: we have two linear maps and a symmetric bilinear one, which differ from ordinary linear and bilinear forms only in that they are not real but one-dimensional linear space valued. We shall call such forms generalized forms.
Another (equivalent) way to describe the situation is the following: the decomposition of V determines 1-dimensional subspaces in $Hom(V, R)$ and in $Hom(M_0, R)$ and a half of a linear space in $Hom(Sym^2(E_0), R)$ (this half is the positive definite half), in other words we have linear forms determined up to scalars, i.e we have a kind of conformal structure on V. (The mentioned spaces are just $[kg]^{-1}, [kgs]^{-1}, [kgm]^{-2}$, they are embedded into $Hom(V, R)$, $Hom(M_0, R)$ and $Hom(Sym(E_0), R)$ by the transpose maps of m, mt and g.)

Note, that every inertial reference frame induces linear isomorphisms of $[kg]$, $[kgm]$ and $[kgs]$ with R. These isomorphisms are the same for two inertial reference frames which are related by a map in GP^R.
All spaces obtained from V by multiplication or division by some power of $[kg]$, $[kgm]$ or $[kgs]$ have a unique Newtonian structure on them isomorphic to the one on V up to real or positive real numbers, and an inertial reference frame of V determines unique inertial reference frames on them.

Let us now introduce the following linear spaces:

`2.7. DEFINITION:`
$[m] := |([kgm]/[kg])|$,
$[s] := [kgs]/[kg]$.
We call $[kg]$ the measure line of mass,
$[m]$ the measure line of distance and
$[s]$ the measure line of periods of time. $\diamond$

The notation $[m]$ comes from 'meter', $[s]$ comes from 'second' and $[kgm]$ comes from 'kilogram'.

We specify now some groups related to G:

`2.8. DEFINITION:`
— $\mathcal{T}^4$: group of parallel translations, the kernel of the homomorphism $G \to G\lceil M_0{}^*$, a 4-dimensional Abelian Lie group, canonically isomorphic to the additive group of $M_0/[kg]$,
— $SO\mathcal{B} \equiv G\lceil M_0$: homogenous Galilei group, 6-dimensional,
— $\mathcal{B}$: group of velocity transformations (i.e. Galilei boosts), the kernel of the homomorphism $G\lceil M_0 \to G\lceil E_0$. It is a 3-dimensional Abelian Lie group, canonically isomorphic to the additive group of $E_0/[kgs]$,
— $SO(g) \equiv G\lceil E_0$: group of rotations, a simple Lie group (isomorphic to $SO(3)$),

* $\lceil$ denotes the restriction of a map.

— $\mathcal{BT}^4$: the kernel of the homomorphism $G \to G\lceil E_0$, nilpotent 7-dimensional Lie group,
— $\mathcal{T}^3 \equiv E_0/[kg]$: group of spacelike translations, canonically isomorphic to the additive group of $E_0/[kg]$. $\diamond$

Consider now the following figures and sets:

`2.9. DEFINITION:`
$M = \{$ lines in V containing 0 and not lying in $M_0\}$,
$T = \{$ 4-dimensional subspaces in V containing $E_0\} \setminus M_0$,
$E^3(t) = \{e \in M | e \subset t\}$ for all $t \in T$.
$\bar{G}$ acts on both M and T. $\diamond$

`2.10. PROPOSITION:` M is a 4-dimensional affine space over $M_0/[kg]$, T is a 1-dimensional affine space over $[s]$. G acts on T by the additive group of $[s]$ effectively. $\diamond$

`2.11. DEFINITION:`
We denote the additive group of $[s]$ by $\mathcal{T}^1$ and call it the group of time translations. $\diamond$

`2.12. DEFINITION:`
We denote the kernel of the homomorphism $G \to \mathcal{T}^1$ by $SO\mathcal{BT}^3$. Let $\mathcal{BT}^3 = SO\mathcal{BT}^3 \cap \mathcal{BT}^4$. $\diamond$

`2.13. PROPOSITION:` (The normal subgroups of G) We have all normal Lie subgroups of G now, these are: $\mathcal{T}^3$, $\mathcal{T}^4$, $\mathcal{BT}^3$, $\mathcal{BT}^4$, $SO\mathcal{BT}^3$. The corresponding factor groups are: $SO\mathcal{BT}^1$, $SO\mathcal{B}$, $SO\mathcal{T}^1$, $SO(g)$, $\mathcal{T}^1$. In figure 1. and 2. we can see the net of normal Lie subgroups of G and the dual net of factor groups. The arrows indicate canonical inclusions and canonical homomorphisms. $\diamond$

For all $t \in T$ the $E^3(t)$ is an affine subspace of M over $E_0/[kg]$, so there is a (conformal) Euclidean structure on every $E^3(t)$.

We introduce the following maps and notations:

`2.14. DEFINITION:`
$u : V \setminus M_0 \to M,\ p \mapsto \{$ the line in M containing $p\}$
$\tau : V \setminus M_0 \to T,\ p \mapsto \{$ the 4-dimensional subspace in T containing $p\}$
$d_t : E^3(t) \times E^3(t) \to [m], \quad (x, y) \mapsto ||x - y|| = md(x - y).$

Note, that $u(p) = p/m(p)$. M is canonically embedded into $V/[kg]$ and as a subset it is determined by the property $(m/[kg])(M) = 1$. $\diamond$

We introduce now the following names:

m : mass (evaluation function)

M : Galilei space-time, the elements of which are called events,
T : time line, the elements of which are called points of time,
$E^3(t)$: synchronous space at $t \in T$,
τ : time (evaluation function),
u : place (in the space-time, evaluation function),
d_t : synchronous distance function at $t \in T$.

M with τ and the d_t-s (or equivalently M with the affine G-action on it) forms a usual Galilei spacetime, which was introduced by H. Weyl [1] (see also [2], [3], [4]). Note, that τ is a synchronization of M and makes M a bundle over T.
Note also that $V \setminus M_0 \ = \ M \times ([kg] \setminus \{0\})$, where the projections are m and u.

`2.15. DEFINITION:`
We call (V, G) an oriented Newtonian space, if its structure is supplemented by an orientation of $[kg]$, $[kgs]$ and E_0. (This has a smaller automorphism group.) $\diamond$

If a Newtonian space is oriented, we can speak of future and past, positive and negative mass, and vectorial product in E_0, for example.

3. The Newtonian particle; momentum and force

`2.16. DEFINITION:`
A Newtonian pointlike particle is a function $f : I \to V$, which has the following properties:
— I is a closed interval of T,
— $m \circ f$ is a constant function, this constant is called the mass of the particle (and denoted by $m(f)$),
— $t \circ f = id_I$ (i.e. f is a natural parametrization of its range). $\diamond$

One can define the category of Newtonian particles, the objects of which are the triplets (V, G, f), where (V, G) is a Newtonian space. A particle can be regarded as an additional structure on a Newtonian space.

`2.17. DEFINITION:` We call $p = f'$ the four-momentum function of f.
Two properties of p are: $Ran(p) \subset M_0/[s]$, $\ \ mt \circ p \equiv m(f)$

We call $v = p/m(f) = (u \circ f)'$ the four-velocity function of f. It satisfies the following: $Ran(v) \subset M_0/[kgs]$, $\ \ mt \circ v \equiv 1$, $\ \ p = m(f) \cdot v$.

We call $F = f''$ the force acting on the particle f. It has the following properties: $RanF \subset E^3/[s]^2, F = m(f) \cdot v' = m(f) \cdot (u \circ f)'$.

We call $a = F/m(f) = v' = (u \circ f)''$ the acceleration function of the particle. For the acceleration and force we have now

$$F = m(f)a = p'$$

$\diamond$

Note, that in general it is $SO\mathcal{BT}^1$ which acts on the four-velocity functions and four-momentum functions, and it is $SO\mathcal{T}^1$ generally which acts on forces and acceleration functions. $SO\mathcal{B}$ acts on the range of four-velocity functions and $SO(g)$ acts on the range of forces.

4. Force field, equation of motion for one body

`2.18. DEFINITION:`
A function $F : V \times M_0/[s] \to E^3/[s]^2$ with an open domain* is called a force field. The differential equation

$$f'' = F \circ (f, f')$$

for a particle f is called Newton's equation of motion, and the solutions of this equation are called the particles determined by the force field in question. An initial value must satisfy the following: $m(f(t_0)) = m_0 \neq 0,\ m_0 = mt(f'(t_0))$. $\diamond$

One can define the category of Newtonian force fields or Newtonian mechanical systems (the objects of which are the triplets (V, G, F)). The automorphism group of such an object can be called the dynamical symmetry of the system. The only force field which is invariant under the Galilei group is the zero field, which determines free particles.

`2.19. DEFINITION:`
Let $f_i, i = 1..N$ be N particles. We introduce their
center of mass : $f = \sum_{n=1}^N f_i$,
total momentum : f',
internal angular momentum : $J = \sum_{i<j} \frac{m_j f_i - m_i f_j}{m_i + m_j} \wedge (\frac{f_i}{m_i} - \frac{f_j}{m_j})'$. $\diamond$

5. The space of four-velocities

`2.20. DEFINITION :`
Let $V(1) = \{$ the lines in M_0 containing 0, not lying in E_0 $\} \equiv \{ v \in M_0/[kgs],$
$mt(v) = 1 \}$.
$V(1)$ is the space from which the four-velocity functions take their values. $\diamond$

`2.21. PROPOSITION :`
$V(1)$ is a 3-dimensional Euclidean affine space over $E_0/[kgs]$.
$M_0 \setminus E_0 = V(1) \times ([kgs] \setminus \{0\})$, where the projections are id_{M_0}/mt and mt.
$\diamond$

6. Effects of fixing a reference frame

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* This is not the weakest satisfactory condition for the domain.

Now we mention a few important additional structures on (V, G) which arise when a particular inertial reference frame is chosen.

A certain timelike line in M is determined, which is called the origin (of the reference frame), E_0 gets an oriented orthogonal basis and the three axes of space, M gets a direct product structure : $M = (E_0/[kg]) \times [s]$, V_1 gets a zero and is mapped to $E_0/[kgs]$, so the kinetic energy of a particle can be defined, $[kg]$, $[kgs]$, $[kgm]$ get units (and orientation), and the factor groups listed in 2.13. and C obtain well-defined monomorphisms into $\bar{G}$, $\bar{G}$ obtains a parametrization and its Lie-algebra a basis. (Many other things could be mentioned, for a more detailed description see e.g. [4].)

If a mass m, a position three-vector x and a point of time t is given with respect to some reference frame, then the corresponding vector in $R^5 \equiv R^3 \times R \times R$ is

$$(mx, mt, m).$$

The coordinate form of the points of M is

$$(x, t, 1),$$

of the points of $V(1)$ is

$$(v, 1, 0),$$

and the functions m, mt, md, τ, d_t, u have the following forms:

$$m : (mx, mt, m) \mapsto m,$$

$$mt : (mx, mt, 0) \mapsto mt,$$

$$md : (mx, 0, 0) \mapsto ||mx||,$$

$$\tau : (mx, mt, m) \mapsto t,$$

$$d_t : ((m_1x, m_1t, m_1), (m_2y, m_2t, m_2)) \mapsto ||x - y||,$$

$$u : (mx, mt, m) \mapsto (x, t, 1).$$

If $f(t) = (mx(t), mt, m)$ is a particle, then

$$p(t) = (mx'(t), m, 0),$$

$$F(t) = (mx''(t), 0, 0),$$

$$v(t) = (x'(t), 1, 0),$$

$$a(t) = (x''(t), 0, 0).$$

III. SPECIAL RELATIVITY

1. Einsteinian space

3.1. DEFINITION:
We call the following 10-dimensional Lie-group the Poincare group:

$$P^R = \left\{ \begin{pmatrix} L & x \\ 0 & 1 \end{pmatrix} \in GL(R^4 \times R) | L \in SO^+(3,1), x \in R^4 \right\}$$

$\diamond$

We define the category of Einsteinian mechanical spaces by replacing the Galilei group with the Poincare group in Definition 1.1.

2. Properties of the Einsteinian space

For the description of the properties of Einsteinian mechanical spaces we assume, that we are given a certain one: (V, P).

3.2. PROPOSITION:
The group of automorphisms of (R^5, P^R) is a 12-dimensional Lie group:

$$\bar{P}^R = \left\{ \begin{pmatrix} A & a \\ 0 & b \end{pmatrix} \in GL(R^4 \times R) | A = n \cdot L,\ L \in SO^+(3,1), a \in R^4, \right.$$

$$\left. n, b \in R \setminus \{0\} \right\}.$$

The elements of $\bar{P}^R$ are again the transformations between inertial reference frames. Each element $\bar{p}$ of $\bar{P}^R$ can uniquely be written in the form

$$\bar{p} = cp,$$

where $p \in P^R$, $c \in C^R$ and C^R is a subgroup of $\bar{P}^R$:

$$C^R = \left\{ \begin{pmatrix} a \cdot Id & 0 \\ 0 & b \end{pmatrix} \in GL(R^4 \times R\ ) |\ a, b \in R \setminus \{0\} \right\},$$

and $P^R \cap C^R = \{Id_{R^5}\}$. P^R is an invariant subgroup of $\bar{P}^R$, C^R is not. $\diamond$

Denoting the automorphism group of (V, P) by $\bar{P}$ we define $C = \bar{P}/P$, this is isomorphic to C^R.

3.3. PROPOSITION: The following subsets of $R^5 \equiv R^4 \times R$ are invariant under the action of P^R:

— for every $m \in R$ the set $M_m^R = \{(x,z) \in R^5 \mid z = m\}$,
— for every $mt \in R^+$ the set $H_{mt}^R = \{(x,z) \in R^5 \mid z = 0,\ < x, x >= (mt)^2 \cdot (-1),\ \ x_4 > 0\ \}$,
— for every $mt \in R^-$ the set $H_{mt}^R = \{(x,z) \in R^5 \mid z = 0,\ < x, x >= (mt)^2 \cdot (-1),\ \ x_4 < 0\ \}$,
— for every $md \in R^+$ the set $S_{md}^R = \{(x,z) \in R^5 \mid z = 0,\ < x, x >= (md)^2\ \}$,
— the set $L^R = \{(x,z) \in R^5 \mid z = 0,\ < x, x >= 0\ \}$
— the set $S_0^R = \{\ 0\ \}$.

Here $<,>$ is the standard product on R^4 with signature $(+++-)$.

The sets M_m^R are parallel 4-dimensional hyperplanes of R^5,
the sets H_{mt}^R are similar connected components of 3-dimensional two-sheeted hyperquadrics in M_0^R,
the sets S_{md}^R are similar 3-dimensional one-sheeted hyperquadrics in M_0^R,
L^R is the 3-dimensional light cone in M_0^R,
S_0^R is just the 0 in R^5.

The orbits of the action of P^R are:
$M_m^R,\ m \in R \setminus \{0\}$,
$H_{mt}^R,\ mt \in R \setminus \{0\}$,
$S_{md}^R,\ md \in R_0^+$, and
L^R.

Thus V also decomposes uniquely to invariant hyperplanes and hyperquadrics and a cone. This decomposition can be obtained by pulling back the one of R^5 to V by any inertial reference frame. $\diamond$

We introduce the following notation for the components of V:
$M_{m(v)}$ denotes the 4-dimensional hyperplane which contains v,
$H_{mt(v)}$ denotes the connected component of the 3-dimensional two-sheeted hyperquadric which contains v, where $v \in M_0 \equiv M_{m(0)}$,
$S_{md(v)}$ denotes the 3-dimensional one-sheeted hyperquadric which contains v, where $v \in M_0$,
L denotes the light cone,
$S_0 = \{0\}$.

The analogue of Proposition 1.3. holds, so the Einsteinian structure could have been defined in terms of the above decomposition.
We see, that we have a kind of conformal structure again.

Let us denote the set of all 4-dimensional hyperplanes of the decomposition of V by $[kg]$,
the set of the H -s and S_0 by $[kgs]$,
the set of the S -s and S_0 by $[kgm]_0^+$.
$[kg] = V/M_0$, $[kgs]$ is an unoriented 1-dimensional linear space (multiplication and addi-

tion can be defined by using suitable representatives) and $[kgm]_0^+$ is the nonnegative half of an oriented 1-dimensional linear space $[kgm]$. We introduce further notations and maps:

3.4. DEFINITION:
$[m] = |[kgm]/[kg]|$: the measure line of distances,
$[s] = [kgs]/[kg]$: the measure line of time periods,
$[kg]$: the measure line of mass.
$m : V \to [kg],\ v \mapsto M_{m(v)}$
$mt : M_0 \to [kgs],\ v \mapsto H_{mt(v)}$
$md : M_0 \to [kgm]_0^+,\ v \mapsto S_{md(v)}$. $\diamond$

3. The velocity of light

The light cone in M_0 determines Lorentzian quadratic forms on M_0 up to nonzero real factor, so by choosing one of them, say l, we can define two maps, $c_1,\ c_2$ from $[kgm]$ to $[kgs]$ (i.e. two elements of $[kgm]/[kgs]$) as follows:

$$c_1(x) = l^{-1}((-1)\cdot l(x)), \qquad x \in [kgs]_1,$$

$$c_2(x) = l^{-1}((-1)\cdot l(x)), \qquad x \in [kgs]_2,$$

where $[kgs]_1$ and $[kgs]_2$ denote the two halves of $[kgs]$. c_1 and c_2 should be extended linearly to the whole $[kgs]$. Then $c_1 = -c_2$. c_1 and c_2 are independent of the choice of l.

An other way to define these elements of $[m]/[s]$ is to choose a 2-dimensional subspace in M_0 which has nonzero intersection with the elements of $[kgs]$, i.e. Lorentzian. The light cone intersects this plane in two lines (which intersect each other in 0). These lines determine reflections which map the intersections of the elements of $[kgs]$ and $[kgm]^+$ with the plane into each other. Two maps can be obtained in this way (after linear extensions), and they are independent of the choice of the 2-dimensional plane and are identical to c_1 and c_2.

3.5. DEFINITION:
$c = |c_1| = |c_2| \in |[m/s]|$ is called the velocity of light. (This name originates from electrodynamics.) $\diamond$

(The velocity of light is – obviously – invariant under the action of $\bar{P}$.) According to the definitions the velocity of light equals 1 in every inertial reference frames. The vector space $[m]/[s]$ can be identified with R algebraically. The elements of $[m]$ and $[s]$ can be distinguished by their geometrical meaning, however.

We have now the following maps on V:
$m : V \to [kg]$,
$<,>: M_0 \times M_0 \to [kgm] \quad \text{or} \quad |[kgs]|$.

(m is already defined, $<,>$ is the generalized Lorentzian form.)
We shall give a description of the subgroups of P now.

`3.6. PROPOSITION :`
The net of normal Lie subgroups and its dual net is the following: see fig. 3., 4.

$\mathcal{T}^4$ is the kernel of the homomorphism $P \to P\lceil M_0$, equivalent to the additive group of $M_0/[s]$,
$\mathcal{L} = P\lceil M_0$, it is called the (homogenous) Lorentz group.

Other important subgroups of $\mathcal{L}$:
— stabilizers of timelike vectors in M_0 ; these are conjugate subgroups, each of them is the special orthogonal group of the orthogonal space of the stabilized vector,
— stabilizers of spacelike vectors in M_0 ; these are conjugate subgroups isomorphic to $SO(2,1)$, i.e. each of them is the special orthogonal group of the orthogonal space of the stabilized vector,
— stabilizers of lightlike vectors in M_0 ; these are conjugate subgroups, isomorphic to the group (called Euclidean group) :

$$\left\{ \begin{pmatrix} O & 0 \\ a & 1 \end{pmatrix} \in GL(R^2 \times R) | O \in SO(2), a \in R^2, \right\}.$$

(This is the same representation as the one we really have). Each of these groups act on the orthogonal space of the stabilized vector.
— groups of boosts: these are the groups which preserve the orthogonal decompositions $B_1 \oplus B_2$ of M_0, where the components are nonsingular subspaces and the action of which on the spacelike part of the decomposition is the identity. The boosts form conjugate subgroups which are isomorphic to the additive group of R.

`3.7. DEFINITION:`
$M = \{$ the lines in V containing 0 and not lying in M_0 $\}$ $\diamond$

M is a 4-dimensional affine space over $\mathcal{T}^4$, $\mathcal{T}^4 \equiv M_0/[kg]$. The stability groups of the points of M are isomorphic to $\mathcal{L}$.

`3.8. DEFINITION :`
We introduce the following map:
$u : V \setminus M_0 \to M, \quad p \mapsto \{$ the line in M containing $p\}$. $\diamond$

M, being an affine space over $\mathcal{T}^4$, carries a further structure: the distance function

$$d : M \times M \to [m], \quad (a, b) \mapsto ||(a - b)||$$

.

3.9 DEFINITION :
We introduce the following names

m : mass (evaluation function)
M : spacetime
u : place (in spacetime, evaluation function)
d : distance (in spacetime). $\diamond$

P acts on M (obviously). M with the affine structure and distance map (or equivalently with the action of P) is a relativistic space-time or a Minkowskian space (not vector space) in the usual sense. The elements of M are called events. $\diamond$

The spaces M_m, $\ m \in [kg]$, $m \neq 0$ are Minkowskian spaces as well, and their isomorphisms with each other arise from the action of the elements of $\bar{P}$ on V. (Note that the distance function is not real valued here, but the elements of $\bar{P}$ do act on its values nontrivially!) In other words, $d_{m_2} \circ p \times p = \tilde{p} \circ d_{m_1}$, where $p \in \bar{P}$, $p(M_{m_1}) = M_{m_2}$ and $d_m = m \cdot d$.

4. Particles, momentum, velocity, force

3.10. DEFINITION:
An Einsteinian pointlike body is described by a function $f : I \to V$ which satisfies the following:
— I is a closed interval of $[s]$,
— it is contained by some M_m for some $m \in [kg], m \neq 0$, which is called the mass of f, denoted by $(m(f))$,
— $||\dot{f}|| \equiv m(f)$, i.e. the range of f is naturally parametrized. $\diamond$

One can define the category of Einsteinian particles.

3.11. DEFINITION:
For two points $a, b \in I$ we call $|a-b| \in |[s]|$ the proper time passed between the two points $f(a)$, $f(b)$ along the path of the particle f. $\diamond$

3.12. DEFINITION:
Let f be a particle. We call $p = f'$ the four-momentum of the particle,
$v = (u \circ f)' = 1/m(f) \cdot p$ the four-velocity of the particle,
$F = f''$ the four-force acting on the particle,
$a = 1/m(f) \cdot f'' = (u \circ f)''$ the four-acceleration of the particle.
Note, that $p = m(f)v$,

$$F = m(f)a,$$

and $||v|| = 1$, $< a, v >= 0$ holds. $||p|| = m(f) \cdot c$, $||p||^2/m(f) = m(f)c^2$ is called the rest energy of f. $\diamond$

5. Force field, equation of motion

`3.13. DEFINITION:`
We call a map $F : V \times M_0 \to M_0/([s]^2)$ which satisfies $< F(x,p), p >= 0$ a force field.

A particle f is said to be determined by the force field F if

$$F \circ (f, f') = f''.$$

$\diamond$

One can define the category of Einsteinian force fields.

6. Space of four-velocities

The space of four-velocities is $V(1) = \{$ the timelike lines containing 0 in M_0 $\} \equiv \{v \in M_0/[kgs] \mid ||v|| = 1$ $\}$, which is (by definition) the 3-dimensional hyperbolic space.

`3.14. DEFINITION:`
Let us choose a spacelike 3-dimensional subspace in M_0, denoted by E. This determines two orthogonal projections: P_E onto E and $P_E^{\perp}$ onto $E^{\perp}$. Using these projections we define the (bijective) map (which we call Cayley map):

$$\Gamma_E : M_0 \to E/|[kgs]|, \quad v \mapsto P_E(v)/|P_E^{\perp}(v)|, \quad \Gamma_E(0) = 0.$$

Γ_E maps $V(1)$ into $E/|[kgs]|$, onto the open ball $\{v \in E/|[kgs]| \mid ||v|| < c\}$. The pair $(E/|[kgs]|, \Gamma_E)$ is known as the Beltrami - Cayley - Klein model of the hyperbolic space. $(E/|[kgs]|$ is a hyperbolic space with the Riemannian metric on it which is pushed forward onto it from $V(1)$ by Γ_E). $\diamond$

`NOTE:` Clearly this model carries more structure than a hyperbolic space. P_E defines another model of the hyperbolic space. The choice of E also determines certain boosts and the stabilizer subgroup of E in the Lorentz group.

7. Effects of fixing a reference frame

The choice of an inertial reference frame also determines a Cayley map, and when one speaks of three-velocities in special relativity then it is just the image of a four-velocity by a Cayley map. On the other hand, three-momenta are obtained from four-momenta by a P_E (and the energy component is obtained by the $P_E^{\perp}$).
An inertial reference frame also determines an origin, units in $[kg]$ and $[kgs]$, a parametrization of $\bar{P}$, a certain $SO(3)$ subgroup and certain boosts in $\bar{P}$ and a basis of the Lie-algebra of P.

If a mass m, and a position vector x^μ is given in some reference frame, then the corresponding vector in $R^5 \equiv R^4 \times R$ is

$$(mx^\mu, m).$$

The coordinate form of the points of M is

$$(x^\mu, 1),$$

of the points of $V(1)$ is

$$(v^\mu, 0),$$

and the functions m, mt, md, d, u have the following forms:

$$m : (mx^\mu, m) \mapsto m,$$

$$mt : (mx^\mu, 0) \mapsto m\sqrt{(-1)x^\mu x_\mu},$$

$$md : (mx^\mu, 0) \mapsto m\sqrt{x^\mu x_\mu},$$

$$u : (mx^\mu, m) \mapsto (x^\mu, 1).$$

If $f(t) = (mx^\mu(\tau), m)$ is a particle, then

$$p(\tau) = (mx'^\mu(\tau), 0),$$

$$F(\tau) = (mx''^\mu(\tau), 0),$$

$$v(\tau) = (x'^\mu(\tau), 0),$$

$$a(\tau) = (x''^\mu(\tau), 0).$$

Furthermore

$$P_E : (x, t) \mapsto (x, 0),$$

$$\Gamma_E : (x, t) \mapsto x/t.$$

IV. OLD RESULTS ABOUT THE ROLE OF MASS

So far we defined the mass as something connected with the action of the Galilei or Poincare group on a linear space. The mass as a quantity connected to the Galilei or Poincare group can be introduced in other settings as well. In this section we wish to display that our approach is in accordance with those in the literature.

According to the result of Wigner and Bargmann the free particles of quantum physics can be brought into correspondence with the elements of certain irreducible ray representations of the Galilei and Poincare group. These representations are naturally parametrized by two quantities: $m \in R_0^+$ and $s \in N/2$ (integers and half-integers) which are identified with the mass (really rest energy in the relativistic case), and spin of the particle (so mass and spin are parameters). (In the case of the Poincare group these parameters are the eigenvalues of the two Casimir operators.)
The analogous result in classical Hamiltonian dynamics is that some of the transitive symplectic representations of the Galilei/Poincare group can be naturally parametrized by a positive real number and these representations can be brought into correspondence with the free particles. The number is again identified with the mass of the particle.
We recall now these theorems and then we formulate them within the framework of our setting. This is done to show that if we formulate these theorems in our setting, then the mass defined by us gets into the role of the quantity which is usually called the mass in the context of these theorems.

4.1. THEOREM: (see [5]) Let $R^3 \times R$ be equipped with the Galilei space-time structure (in the standard way). The timelike lines (which are the possible trajectories of a free particle) in $R^3 \times R$ form a 6-dimensional manifold which can be identified with $R^3 \times R^3$. The Galilei group acts on this manifold transitively. Introducing the symplectic form $[(v_1, q_1), (v_2, q_2)] \mapsto < mv_1, q_2 > - < mv_2, q_1 >,\ m \in R^+$ the representation of the Galilei group turns into a transitive symplectic representation of the Galilei group. Two such representations with m_1 and m_2 are equivalent if and only if $m_1 = m_2$.
(By equivalence of two symplectic representations we mean the existence of a G^R-equivariant diffeomorphism ϕ for which $\phi_* \omega_1 = \omega_2$.) $\diamond$

4.2. THEOREM: (see [6]) Let X_H be a Galilei invariant Hamiltonian field on T^*R^3. Then there exists a unique constant $m_0 > 0$ so that X_H corresponds to a free particle of mass m_0. $\diamond$

(X_H is called Galilei invariant, if there is an action of G^R on T^*R^3, realized by symplectic diffeomorphisms so that
the space translations are represented by $(p, x) \mapsto (p, x + a)$,
the rotations are represented by $(p, x) \mapsto (Rp, Rx)$,
time translations are generated by X_H.)

(A free particle of mass $m_0 > 0$ is defined by the Hamiltonian $H = ||x||^2/(2m_0) +$

$const.$, which is Galilei invariant, if the velocity transformations are represented by $(p, x) \mapsto (p - m_0 v, x)$.

Relativistic case:

4.3. THEOREM: (see [5]) Let us denote the space of timelike lines in the Minkowski space $R^3_1 = R^3 \times R$ by M. This is a 6-dimensional manifold which can be identified with $V(1) \times R^3$. Using the fact that this is a submanifold of $R^3_1 \times R^3_1$ we can introduce the following symplectic form: $[(v_1, q_1), (v_2, q_2)] \mapsto < v_1, mq_2 > - < v_2, mq_1 >$, ($<, >$ is the Lorentzian form) with which M carries a transitive symplectic representation of the Poincare group. Two such representations are equivalent if and only if $m_1 = m_2$. $\diamond$

4.4. THEOREM: (see [6]) Let X_H be Poincare invariant. Then there exists a unique constant m so that $H(p, q) = \sqrt{m^2 + ||p||^2} + const.$ $\diamond$

Poincare invariance means here that there is a symplectic representation of the Poincare group on T^*R^3 so that the translations are represented by $(x, p) \mapsto (x + a, p)$, the rotations are represented by $(x, p) \mapsto (Rx, Rp)$ and the time translations are generated by X_H.

The following theorems are reformulated versions of the above ones:

Nonrelativistic case:

4.5. THEOREM: Let us denote the set of the timelike lines in M_m, $m \neq 0$ by $\mathcal{F}_m$. This is a 6-dimensional manifold on which G acts transitively. The map $\pi : \mathcal{F}_m \to V(1)$ which assigns to every element of $\mathcal{F}$ its velocity makes $\mathcal{F}_m$ a bundle over $V(1)$. Each fiber is an affine space over E_0. The tangent space $T_x\mathcal{F}_m$ is $(E_0/[kgs]) \times E_0$ for all $x \in \mathcal{F}_m$. So $\mathcal{F}_m$ has the canonical (generalized) symplectic form : $[(v_1, mq_1), (v_2, mq_2)] \mapsto < v_1, mq_2 > - < v_2, mq_1 >$. With this symplectic structure the action of G on $\mathcal{F}_m$ is symplectic.
(Generalized means not real but 1-dimensional vector space valued.)

Concerning the relationship between $\mathcal{F}$-s with various masses we can say that the following diagram is commutative: see figure 5.
Here g and $\tilde{g}$ are the action of an element of $\bar{G}$ on the corresponding spaces. In particular, two symplectic representations with m_1 and m_2 are equivalent iff $m_1 = \pm m_2$.

Choose an inertial reference frame. This determines symplectic isomorphisms between $(E_0/[kgs]) \times E_0$ and the $\mathcal{F}_m$-s. (And pushes forward the transitive symplectic representation of $\mathcal{F}_m$.) Let $H_m : E_0/[kgs] \times E_0 \to [kg][m]^2/[s]^2$, $(v, mq) \mapsto 1/2 \cdot m||v||^2$. Then X_{H_m} generates the time translations parallel to the fourth axis. $\diamond$

Relativistic case:

`4.6. THEOREM:` Let us denote the set of timelike lines in M_m by $\mathcal{F}_m$. Again there is a natural bundle structure $\pi : \mathcal{F}_m \to V(1)$. The fiber of the bundle over each point v of $V(1)$ is an affine space over the orthogonal of v in M_0. P acts on $\mathcal{F}_m$ transitively. The tangent space $T_x\mathcal{F}_m$ for all $x \in \mathcal{F}_m$ is $(\pi(x)^\perp) \times (\pi(x))^\perp \otimes [kgs])$ for all $x \in \mathcal{F}_m$. So we can fix the (generalized) symplectic structure $((v_1, mq_1), (v_2, mq_2)) \mapsto < v_1, mq_2 > - < v_2, mq_1 >$ which makes the action of P symplectic.
Concerning the relationship between $\mathcal{F}$-s with various m-s we can say that the following diagram is commutative: see figure 5.
Here g and $\tilde{g}$ are the action of en element of $\bar{P}$ on the corresponding spaces. In particular, two symplectic representations with m_1, m_2 are equivalent iff $m_1 = \pm m_2$.

Now choose an inertial reference frame, let x be the unit vector in M_0 parallel with the fourth (time) axis. This reference frame determines symplectic(!) diffeomorphisms $\ell_m : \mathcal{F}_m \to (x^\perp) \times (x^\perp \otimes [kgs])$, where the symplectic structure on $(x^\perp) \times (x^\perp \otimes [kgs])$ is the canonical one and x is the direction of the fourth (time) axis. Thus P acts on $(x^\perp) \times (x^\perp \otimes [kgs])$ transitively by symplectic transformations. Let $X_{H_m} : (v, mq) \mapsto \sqrt{m^2c^4 + m^2||v||^4}$. Then X_{H_m} generates the time translations parallel to the fourth (time) axis. $\diamond$

(To prove that the ℓ_m-s are symplectic take the coordinates determined by the chosen inertial reference frame. Then it turns out, that $\mathcal{F}_m$ is isomorphic (as bundle and symplectic manifold) to $T^*V(1)$, and of course $(x^\perp) \times (x^\perp \otimes [kgs])$ is isomorphic to $T^*(x^\perp)$. Now, composed with these isomorphisms the ℓ_m-s turn into the cotangent map of the projection P_E, where $E = (x)^\perp$). $\diamond$

Conclusion

Our discussion covers the content of Newton's first and second law. The statement of the third and fourth law would be straightforward now.

In our setting the notion of mass, force and momentum, which belong to dynamics conventionally, have become geometrical ones. It is the force field which can be regarded as a proper dynamical notion. We can say, that Newton's first and second law is a specification of certain geometrical circumstances on the one hand, and (a part of) a definition of dynamics on the other hand.
We can see, that in our formulation we didn't need orientations, which displays the known fact that Newtonian and special relativistic mechanics do not have much to do with the orientation of space, time and mass (mathematically). It should be noted, however, that the presence of particles or a force field can easily determine an orientation of the mechanical spaces.
We can also see that Newtonian mechanics does not contain any natural units of length, time and mass. Units are brought into Newtonian mechanics by force fields and particles. Other branches of physics do contain fundamental dimensional constants, of course.

Relativistic mechanics is an example, and here the role of the velocity of light is well understood.
Finally, one should note, (and this is an important point) that while in the old results the notion of mass is associated with free particles, this is not at all the case in our setting.

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APPENDIX

OPERATIONS WITH ONE-DIMENSIONAL LINEAR SPACES

Every linear space which occurs in this section is meant to be real.

`A.1. DEFINITION:` (Division)
Let W be an arbitrary linear space and D a 1-dimensional one.
We call $W/D := Hom(D, W) \equiv W \otimes D^*$ the quotient of W and D. (Care must be taken with this notation, it can be mixed up with the notation of quotient space.) For $w \in W$, $d \in D$ we call the element w/d of $Hom(D, V)$ the quotient of w and d, where w/d is determined by the property $(w/d)(d) = w$. $\diamond$

By product of vector spaces we mean tensorial product. One easily verifies, that the usual identities of multiplication and division hold for the division and multiplication introduced now, i.e.:

$$D \otimes (V/D) \equiv (V/D) \otimes D \equiv V, \quad (V/A)/B \equiv V/(A \otimes B), \quad etc.$$

Note, that the linear spaces obtained from W by multiplication or division by other 1-dimensional linear spaces are not canonically isomorphic, but the isomorphisms between them are determined up to nonzero real numbers (up to positive real numbers in the case when the 1-dimensional space is oriented). Thus the notion parallelism is meaningful regarding the elements from these linear spaces.

`A.2. DEFINITION:`
Let W and X be arbitrary linear spaces and A an 1-dimensional one. Given a linear map $L : W \to X$, this determines the linear maps $L/A \; : \; W/A \to X/A, \quad w/a \mapsto x/a$ and $L \otimes A \; : \; W \otimes A \to X \otimes A, \quad w \otimes a \mapsto x \otimes a$, where a is a nonzero element of A and the defined maps are independent of it. L/A and $L \otimes A$ will be denoted by L, too, for the sake of brevity. $\diamond$

The above notion of quotient of linear spaces was first used in the monograph [4] on the structure of space-time in which it is stated that when we have dimensional quantities rather than bare numbers in physics then we treat 1-dimensional linear spaces in fact. Our treatment of the question of dimensions differs from that in [4], in particular, regarding the question of the role of dimensions in Newtonian mechanics and special relativity.
Given two 1-dimensional linear spaces A and B, the spaces A/B and $A \otimes B$ are oriented if and only if A and B are both oriented. (Similar statement is true when the dimension of A is odd, and B may be unoriented if the dimension of A is even.) On the other hand, the even powers of A : $A^2, A^4, ...$ are oriented anyway, the odd powers are oriented if and only if A is oriented.

`A.3. DEFINITION:`
The factor space of a 1-dimensional linear space D with respect to multiplication by (-1)

is the nonnegative part of a 1-dimensional oriented linear space which we call the absolute value of D and denote it by $|D|$. We call the factor map $|\ | : D \to |D|,\ d \mapsto \{d, -d\}$ absolute value function. $\diamond$

For D being oriented and being canonically isomorphic to $|D|$ are the same thing. We can now introduce arbitrary rational powers of a 1-dimensional real linear space and of the elements of it.

`A.4. DEFINITION:`

Let A be an oriented 1-dimensional linear space. We call a pair (B, i) of an oriented 1-dimensional linear space B and an orientation preserving linear isomorphism i between B^n and A an n-th root of A (n is a positive even number). As any two n-th root of A are canonically isomorphic, we speak of *the* n-th root of A and denote it by $\sqrt[n]{A}$. We call the map $p^{-1} \circ i^{-1}$ the extraction of root, where i is the linear isomorphism between $(\sqrt[n]{A})^n$ and A, and $p : a \mapsto a^n$. For an unoriented 1-dimensional linear space D we define the n-th root as the n-th root of $|D|$. $\diamond$

(One can find explicit realization for the n-th roots of a one-dimensional linear space.)

The definition of a proper rational (i.e. non-integer) power of D is now clear.

REFERENCES

[1] H, Weyl, Space-Time-Matter, Dover publ., 1922
[2] V. I. Arnold, Mathematical methods of classical mechanics, Springer, 1989
[3] A. Prastaro, Geometry of PDEs and Mechanics, World Scientific Publishing Co. Pte. Ltd., 1996
[4] T. Matolcsi, A Concept of Mathematical Physics, Models for Spacetime, Akadémiai Kiadó, Budapest, 1984
[5] J. M. Souriau, Structure des systemes dynamique, Dunod, Paris, 1970
[6] R. Abraham, J. E. Marsden, Foundations of Mechanics, The Benjamin/Cummings Publishing Co., Inc., 1978

figure 1.

$$
\begin{tikzcd}
G & \mathcal{BT}^4 \arrow[l] & \mathcal{T}^4 \arrow[l] \\
SO\mathcal{BT}^3 \arrow[u] & \mathcal{BT}^3 \arrow[l] \arrow[u] & \mathcal{T}^3 \arrow[l] \arrow[u] \\
 & & e \arrow[u]
\end{tikzcd}
$$

figure 2.

$$
\begin{tikzcd}
G \arrow[d] & & \\
SO\mathcal{BT}^1 \arrow[r] \arrow[d] & SO\mathcal{T}^1 \arrow[r] \arrow[d] & \mathcal{T}^1 \arrow[d] \\
SO\mathcal{B} \arrow[r] & SO(g) \arrow[r] & e
\end{tikzcd}
$$

figure 3.

$$
\begin{tikzcd}
P & \mathcal{T}^4 \arrow[l] & e \arrow[l]
\end{tikzcd}
$$

figure 4.

$$P \longrightarrow \mathcal{L} \longleftarrow e$$

figure 5.

$$\begin{tikzcd}
T^2\mathcal{F}_{m_1} \arrow[r, "Tg\otimes Tg"] \arrow[d, "\omega_1"'] & T^2\mathcal{F}_{m_2} \arrow[d, "\omega_2"'] \\
{[kgm][m]/[s]} \arrow[r, "\tilde{g}"] & {[kgm][m]/[s]}
\end{tikzcd}$$